

Unknown single oscillator coherent states do have statistical significance.

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It is shown, contrary to popular belief, that *single unknown* oscillator coherent states can be endowed with a *measurable statistical significance*.

I. INTRODUCTION

Nature of single quantum states has been the subject of a lot of debate right from the early days of Quantum theory and has a very important role in the interpretation of quantum theory. According to the currently accepted picture, an unknown single quantum state can not be endowed with any *statistical significance* and as per the *ensemble interpretation* of quantum mechanics is not physically meaningful. Though proposals like *Protective Measurements*[2] claiming to provide a non-destructive measurement on certain classes of unknown single states and yet providing full information about the state have been made, a more careful examination [3] has revealed that even very mild departures from Adiabaticity, as indeed would be in any realistic measurement process, upsets the *Protection* in an uncontrollable manner. See however [4] for a pragmatic use of the concept of Protective Measurements.

In fact this interpretation of quantum mechanics asserts that only *ensembles of identically prepared states* are meaningful in quantum theory in the sense that only measurements on such ensembles yield unambiguous results. This has to do with the way the act of measurement is interpreted in quantum theory. According to this, if an *observable* \mathcal{A} is measured in any state which is not its *eigenstate*, the outcome of any *single measurement* is a *purely random* choice of the *eigenvalues* of \mathcal{A} . Further, the state after the measurement becomes the corresponding eigenstate irrespective of the original state.

It is clear that such an interpretation of the act of measurement introduces a dramatic difference between the cases where the measurement is done on *unknown* and *known* states. When the original state is known, one has the *option* of making the measurement of a suitable observable of which the known state is an eigenstate. Such a measurement has the following features: i) it *unambiguously* yields the eigenvalue of the observable being measured, ii) it does not alter the original state and iii) it allows all compatible (mutually commuting) observables to also be measured without any ambiguity.

In contrast, if the original state is *unknown*, generically any observable chosen to be measured in this state would not be such that the unknown state is its eigenstate. Then as per Quantum Measurement theory, the outcome can be any of the eigenvalues of the observable under measurement and it would be impossible to predict which of the eigenvalues will be the outcome in any given measurement. Further, since the state after measurement changes to the corresponding eigenstate, the

original state is *irretrievably altered*. Even if *accidentally* the observable happens to be such that the original state is its eigenstate, it will not be possible to interpret the result of the outcome without the explicit knowledge about this. Since the state has been changed after the measurement, repeated measurements subsequently yield no *information* about the original state.

It is of course possible to give a *Bayesian estimate* for the unknown state based on the outcome of the single measurement, but there is no way of either confirming or improving this estimate with subsequent measurements as the state after the first measurement is not *correlated* with the initial unknown state.

The well known *no cloning theorem* [5] which asserts that in quantum theory it is *impossible* to make copies of an *unknown* single state, in fact provides a remarkable consistency to the abovementioned interpretation. For, if such a copying were possible, one could have created an *arbitrarily large* ensemble and determined the statistical significance through *ensemble measurements*. It is often stated that *orthogonal* states can be copied but it should be stressed that even that is possible only *when the orthogonal family* is known beforehand.

Motivated by the no cloning theorem there is a vast literature on the so-called *optimal cloning* [6–8]. In these implementations, one starts with the *original unknown state* $|\alpha\rangle$ belonging to the Hilbert space \mathcal{H}_A , a number of *blank states* $|b_0\rangle, |b_1\rangle, \dots, |b_N\rangle$ belonging respectively to the Hilbert spaces \mathcal{H}_{B_i} each of which is isomorphic to \mathcal{H}_A and a number of *machine states* $|m_0\rangle, |m_1\rangle, \dots, |m_M\rangle$ belonging to the Hilbert space \mathcal{H}_M . The combined Hilbert space has the structure $\mathcal{H}_A \otimes \mathcal{H}_M \otimes \prod_i \mathcal{H}_{B_i}$. Then the *optimal cloning transformation* \mathcal{T} has the effect

$$|\alpha\rangle \prod_0^N |b_i\rangle \prod_0^M |m_j\rangle \xrightarrow{\mathcal{T}} \sum_{\{i,j,k\}} d_{\{ijk\}} \prod_i |a_i\rangle \prod_j |b_j\rangle \prod_k |c_k\rangle \quad (1)$$

in such a way that *all* the reduced density matrices ρ_{i_0} obtained by tracing over the \mathcal{H}_A states, the machine states and all the blank states except those belonging to $\mathcal{H}_{B_{i_0}}$, are all *identical* and with maximum overlap with the original unknown state $|\alpha\rangle$ i.e with the maximum possible value of $\langle \alpha | \rho_{i_0} | \alpha \rangle$. The reduced density matrices are *mixed*.

It should be noted that at any given time it is not possible to realise more than one of the reduced matrices ρ_i as different values of i require tracing over different states. This means that we can not use these optimal clonings to get any statistical information about the original state.

In fact unitarity precludes getting the final density matrix for all the blank states to be of the form $\rho \otimes \rho \otimes \rho \dots$ etc, which is just the content of the no-cloning theorem.

In this paper we wish to show that by using the concept of *information cloning* proposed by us [9] it is indeed possible to get statistical information about *single unknown coherent states* though a certain price has to be paid for this which will be explained later. In the case of coherent states, complete information about the state is contained in the complex coherency parameter α . Thus by information cloning what we mean is the ability to make arbitrary number of copies of coherent states whose coherency parameter is $c(N)\alpha$ where α is the coherency parameter of the unknown coherent state and $c(N)$ is a known constant depending on the number of copies made.

We consider $1 + N$ systems of harmonic oscillators whose creation and annihilation operators are the set $(a, a^\dagger), (b_k, b_k^\dagger)$ (where the index k takes on values $1, \dots, N$) satisfying the commutation relations

$$[a, a^\dagger] = 1; [b_j, b_k^\dagger] = \delta_{jk}; [a, b_k] = 0; [a^\dagger, b_k] = 0 \quad (2)$$

Coherent states parametrised by a complex number are given by

$$|\alpha\rangle = D(\alpha) |0\rangle \quad (3)$$

where $|0\rangle$ is the ground state and the unitary operator $D(\alpha)$ is given by

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \quad (4)$$

Let us consider a *disentangled* set of coherent states $|\alpha\rangle |\beta_1\rangle |\beta_2\rangle \dots |\beta_N\rangle$, where α is *unknown* while β_i are *known* to very high accuracy, and consider the action of the unitary transformation¹

$$U = e^{-t(a^\dagger \otimes \sum_j r_j b_j - a \otimes \sum_j r_j b_j^\dagger)} \quad (5)$$

By an application of the Baker-Campbell-Hausdorff identity and the fact that $U|0\rangle |0\rangle |\beta_1\rangle \dots |\beta_N\rangle = |0\rangle |0\rangle |\beta_1\rangle \dots |\beta_N\rangle$ it is easy to see that the resulting state is also a disentangled set of coherent states expressed by

$$|\alpha'\rangle |\beta'_1\rangle |\beta'_2\rangle \dots |\beta'_N\rangle = U |\alpha\rangle |\beta_1\rangle |\beta_2\rangle \dots |\beta_N\rangle \quad (6)$$

The initial state is

$$|I\rangle = D(\alpha) D(\beta_1) \dots D(\beta_N) |0\rangle |0\rangle |\beta_1\rangle \dots |\beta_N\rangle \quad (7)$$

Defining

$$a(t) = U a U^\dagger \quad b_j(t) = U b_j U^\dagger \quad (8)$$

one easily gets the differential equations

$$\frac{d}{dt} a(t) = - \sum_j r_j b_j(t) \quad \frac{d}{dt} b_j(t) = r_j a(t) \quad (9)$$

The solutions to these eqns are straightforward to find:

$$\begin{aligned} a(t) &= \cos Rt a - \sum_j \frac{r_j}{R} \sin Rt b_j \\ b_j(t) &= \frac{r_j}{R} \sin Rt a + \sum_k M_{jk}(t) b_k \end{aligned} \quad (10)$$

where $R = \sqrt{\sum_j r_j^2}$ and

$$M_{jk} = \delta_{jk} - \frac{r_j r_k}{R^2} (1 - \cos Rt) \quad (11)$$

This transformation induces a transformation on the parameters (α, β_j) which can be represented by the matrix \mathcal{U} i.e $\alpha_a(t) = \mathcal{U}_{ab} \alpha_b$. We have introduced the notation α_a with $a = 1, \dots, N+1$ such that $\alpha_1 = \alpha, \alpha_k = \beta_{k-1} (k \geq 2)$. Then we have

$$\mathcal{U}_{1a} = \left(\cos Rt \quad \frac{r_1}{R} \sin Rt \quad \dots \quad \frac{r_N}{R} \sin Rt \right) \quad (12)$$

$$\mathcal{U}_{ab} = -\frac{r_{a-1}}{R} \sin Rt \delta_{b1} + (1 - \delta_{b1}) M_{a-1, b-1} \quad (13)$$

where eqn (13) is defined for $a \geq 2$. Equivalently

$$\mathcal{U} = \begin{pmatrix} \cos Rt & \frac{r_1}{R} \sin Rt & \dots & \dots & \frac{r_N}{R} \sin Rt \\ -\frac{r_1}{R} \sin Rt & M_{11} & \dots & \dots & M_{1N} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{r_N}{R} \sin Rt & M_{N1} & \dots & \dots & M_{NN} \end{pmatrix} \quad (14)$$

We wish to choose $\{\beta_i, r_i\}$ in such a way that all $\beta_i(t)$ become identical as we want N identical copies. Clearly this is possible only if $r_i = r, \beta_i = \beta$. In that case we have

$$\beta_i(t) = -\frac{\alpha}{\sqrt{N}} \sin Rt + \beta \cos Rt \quad (15)$$

Let us first consider the choice of $\sin Rt = -1$ which gives N copies of the state $|\frac{\alpha}{\sqrt{N}}\rangle$. This is what we called *information cloning* in [9] as the states $|\frac{\alpha}{\sqrt{N}}\rangle$ and $|\alpha\rangle$ have the same information content. This particular choice of Rt will be seen to be optimal in the sense that it gives the least variance in the estimation of α . In this case the value of β is immaterial.

We can now use the N copies of $|\frac{\alpha}{\sqrt{N}}\rangle$ to make *ensemble measurements* to estimate $\frac{\alpha}{\sqrt{N}}$ and consequently α . One can estimate a state arbitrarily accurately by using a *sufficiently large* ensemble. However, in our proposal even though the number of copies N can be arbitrarily large, the coherency parameter given by $\frac{\alpha}{\sqrt{N}}$ becomes *arbitrarily small* while the *uncertainties* in α remain the same as in the original state. This raises the question as to how

¹ The most general transformation would involve complex r_j 's. But this can be reduced to the present form through suitable redefinitions [9]

best the original state can be reconstructed and about the resultant statistical significance of the unknown single coherent state.

On introducing the *Hermitean* momentum and position operators \hat{p}, \hat{x} through

$$\hat{x} = \frac{(a + a^\dagger)}{\sqrt{2}} \quad \hat{p} = \frac{(a - a^\dagger)}{\sqrt{2}i} \quad (16)$$

the *probability distributions* for position and momentum in the coherent state $|\frac{\alpha}{\sqrt{N}}\rangle$ are given by

$$\begin{aligned} |\psi_{clone}(x)|^2 &= \frac{1}{\sqrt{\pi}} e^{-(x - \sqrt{\frac{2}{N}}\alpha_R)^2} \\ |\psi_{clone}(p)|^2 &= \frac{1}{\sqrt{\pi}} e^{-(p - \sqrt{\frac{2}{N}}\alpha_I)^2} \end{aligned} \quad (17)$$

Let us distribute our N -copies into two groups of $N/2$ each and use one to estimate α_R through position measurements and the other to estimate α_I through momentum measurements. Let y_N denote the average value of the position obtained in $N/2$ measurements and let z_N denote the average value of momentum also obtained in $N/2$ measurements. The *central limit theorem* states that the probability distributions for y_N, z_N are given by

$$\begin{aligned} f_x(y_N) &= \sqrt{\frac{N}{2\pi}} e^{-\frac{N}{2}(y_N - \sqrt{\frac{2}{N}}\alpha_R)^2} \\ f_p(z_N) &= \sqrt{\frac{N}{2\pi}} e^{-\frac{N}{2}(z_N - \sqrt{\frac{2}{N}}\alpha_I)^2} \end{aligned} \quad (18)$$

The *estimated* value of α is

$$\alpha_{est} = \frac{\langle y_N + iz_N \rangle}{\sqrt{2}} \sqrt{N} = \alpha \quad (19)$$

Thus the original unknown α is correctly estimated. But this is not enough and one needs to know the reliability of this estimate. For that one needs the variance. The variances in y_N, z_N given by eqn.(18) are

$$\Delta y_N = \frac{1}{\sqrt{N}} = \Delta z_N \quad (20)$$

resulting in the variance for α of

$$\Delta\alpha_R = \Delta\alpha_I = \frac{1}{\sqrt{2}} \quad (21)$$

Thus, while the statistical error in usual measurements goes as $\frac{1}{\sqrt{N}}$, and can be made arbitrarily small by making N large enough, information cloning gives an error that is fixed and equal to the variance associated with the original unknown state.

One practical problem would be due to the fact that $N \rightarrow \infty$ the coherence parameter for the information cloned states becomes very small leading to large noise. This can be circumvented by making a different choice of Rt . For example the choice $\sin Rt = \frac{1}{\sqrt{2}}$ would yield information cloned states with parameter $\frac{\alpha}{\sqrt{2N}} + \frac{\beta}{\sqrt{2}}$. With a large enough β the signal to noise ratio can be improved. further, if the errors in the prior knowledge of β (recall that $|\beta\rangle$ are *known* states) is much better than $\frac{1}{\sqrt{N}}$, the tiny α -dependent part can still be extracted, but the variances will now have increased to $\Delta\alpha_R = \Delta\alpha_I = 1$. There is also a way of tackling the signal to noise problem without significantly compromising the variance. For this choose $\sin Rt$ to be very close to -1, say $-1 + \epsilon$. Then the cloned states parameter will be $\alpha(1 - \epsilon)/\sqrt{N} + \sqrt{2}\epsilon\beta$. Choosing a large enough and accurately known β one can avoid the small signal to noise problem, but the variance will have increased only very slightly by a factor $\frac{1}{1 - \epsilon}$.

Thus we have shown that even when the coherent state is *unknown* single state, information cloning will allow its determination. Of course the statistical errors can not be reduced but it is still a long way from not being able to know anything at all about the unknown state.

II. ACKNOWLEDGEMENTS

The author would like to express his gratitude to the Department of Atomic Energy for the award of a Raja Ramanna Fellowship which made this work possible, and to CHEP, IISc for its invitation to use this Fellowship there.

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